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On the numerical treatment of the Navier-Stokes equations for an incompressible fluid

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SUMMARY
A new approach for the numerical integration of the Navier-Stokes equations is presented. It is based on the perturbation of the Poisson type equation. The system of nonlinear partial differential equations obtained is solved by means of an explicit operator. Some results are presented for the problem of steady flow in a square cavity. They are in good agreement with the calculations made by Greenspan, Fortin, Teman and Peyret, Bourcier and François and Burgraff. Reynolds numbers up to 400 have been considered.

It is concluded that the method is relatively simple and fast in the lower range of Reynolds numbers but the computing time increases very much when higher Reynolds numbers are reached.

1. Introduction

In order to gain more insight on the general problem of atmospheric diffusion a thorough understanding of the basic phenomena of laminar flow of a viscous incompressible fluid is required. These phenomena are described by the complete Navier–Stokes equations. Then, in the first stage, our purpose has been to study the particularities of this problem. Later on successive refinements will include buoyant forces and the diffusion process in itself.

The numerical solution of the Navier–Stokes equations has been tested with the classical problem of the square cavity flow. Several authors have dealt with it and nowadays various numerical techniques and results are available. In this sense our method is one of the closely related convergent methods used on the subject. The relative merit of them has still to be studied.

The present work is divided as follows:

In section 2 the general problem of viscous incompressible flow is formulated and the finite difference analogue approaching it is introduced in section 3. Section 4 is devoted to the formulation of the square cavity flow problem. In section 5 the numerical treatment and in section 6 the prelimininary tests, are given. In section 7 the results obtained for different Reynolds numbers, time-steps and grid sizes are discussed and in section 8 the general conclusions are drawn.

2. Formulation of the general problem

Let us consider the following system representing the Navier–Stokes equations of a laminar viscous incompressible flow and the continuity equation (in two dimensions).

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + \sum\limits_{i=1}^{2} u_i \dfrac{\partial u}{\partial x_i} + \dfrac{1}{\rho} \operatorname{grad} p = 0 & (1.a) \\ \operatorname{div} u = 0\,. & (1.b) \end{cases}$$

where u_i are the velocity components, p is the pressure, ρ is the density and ν is the kinematic viscosity.

Eliminating the variable p we have the so-called transport vorticity equation

$$\frac{\partial \omega}{\partial t} - \nu \Delta \omega + \sum_{i=1}^{2} u_i \frac{\partial \omega}{\partial x_i} = 0, \quad \omega = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}. \tag{2}$$

Introducing the stream function φ, we have

$$\frac{\partial\varphi}{\partial x_2} = u_1, \quad \frac{\partial\varphi}{\partial x_1} = -u_2 .$$

System (2) becomes

$$\begin{cases} \dfrac{\partial\omega}{\partial t} - \nu\Delta\omega + \sum_{i=1}^{2} u_i \dfrac{\partial\omega}{\partial x_i} = 0, \quad i=1,2, & (3.a) \\ \Delta\varphi = -\omega . & (3.b) \end{cases}$$

We are seeking functions $W=\{\omega, \varphi\}$ defined in the cylinder $G=\Omega\times[0, T]$, where Ω is a bounded two-dimensional region, with boundary Γ, $T>0$, and satisfying (3) for:

$$x=\{x_1, x_2\}\in\Omega, \quad 0<t<T ,$$

together with

$$\varphi(x,t)=0, \quad x\in\Gamma, \ t>0,$$
$$\omega(x,0)=\omega_0(x), \quad x\in\Omega ,$$
$$\frac{\partial\varphi}{\partial n} = g(x), \quad x\in\Gamma .$$

Due to equation (3b) system (3) is not a regular system or in other words, it is not a system of Cauchy–Kovaleska type ([1], [3], [4] and [9]).

Let us now associate to problem (3) the following perturbated problem of Cauchy–Kovaleska type

$$\begin{cases} \dfrac{\partial\omega_\varepsilon}{\partial t} - \nu\Delta\omega_\varepsilon + \sum_{i=1}^{2} u_{\varepsilon_i} \dfrac{\partial\omega_\varepsilon}{\partial x_i} = 0. \\ \varepsilon\dfrac{\partial\varphi_\varepsilon}{\partial t} - \Delta\varphi_\varepsilon - \omega_\varepsilon = 0, \quad \varepsilon: \text{small parameter} \end{cases} \tag{4}$$

We are seeking functions $W=\{\omega_\varepsilon, \varphi_\varepsilon\}$ defined in G and satisfying (4) for:

$$x=\{x_1, x_2\}\in\Omega, \quad 0<t<T ,$$

together with

$$\begin{aligned} &\varphi_\varepsilon(x,t)=0, && x\in\Gamma, \quad t>0, \\ &\omega_\varepsilon(x,0)=\omega_0(x), && x\in\Omega, \\ &\varphi_\varepsilon(x,0)=\varphi_0(x), && x\in\Omega, \ \varphi_0(x) \text{ arbitrary}, \\ &\frac{\partial\varphi_\varepsilon}{\partial n} = g(x), && x\in\Gamma. \end{aligned}$$

For a fixed value of ε problem (4) has a unique solution. Furthermore it is conjectured that when $\varepsilon\to 0$, the solution of problem (4) tends to the solution of problem (3).

3. The finite difference analogue

For simplicity we shall write again system (4) in a more condensed form (dimensionless)

$$\frac{\partial W}{\partial t} = -A\frac{\partial W}{\partial x_1} + B\frac{\partial W}{\partial x_2} + C\Delta W + DW \tag{5}$$

where (omitting subindexes)

$$W=\begin{bmatrix}\omega\\ \varphi\end{bmatrix}, \qquad A=\begin{bmatrix}\partial\varphi/\partial x_2 & 0\\ 0 & 0\end{bmatrix}$$

$$B=\begin{bmatrix}\partial\varphi/\partial x_1 & 0\\ 0 & 0\end{bmatrix}, \qquad C=\begin{bmatrix}1/R & 0\\ 0 & c^2\end{bmatrix}, \qquad D=\begin{bmatrix}0 & 0\\ c^2 & 0\end{bmatrix}, \text{ and } c^2=1/\varepsilon, R=UL/\nu .$$

R is the Reynolds number, U is a reference velocity, L is a reference length and ν is the kinematic viscosity.

The finite difference analogue of (5) is written as follows:

$$\frac{W^{n+1}-W^n}{\Delta t} = \left[\frac{-A^n}{2\Delta x}(\Delta_{+1}+\Delta_{-1}) + \frac{B^n}{2\Delta x}(\Delta_{+2}+\Delta_{-2}) + \frac{C}{\Delta x^2}(\Delta_{+1}\Delta_{-1}+\Delta_{+2}\Delta_{-2}) + DE\right] W^n$$

$$+O(\Delta t, \Delta x^2), \Delta x_1 = \Delta x_2 = \Delta x\,, \tag{6}$$

where as usual Δt and Δx indicate the time and space increments, W^n is the discrete value approximating the solution at time $t = n\,\Delta t$, Δ_{+s} and Δ_{-s} are the forward and backward differences, $(s=1, 2)$ and E is the identity operator.

In the following we shall be concerned with the stability problem (more details on the whole problem can be found in [10]).

It is possible to show that a linear stability analysis in the absence of inertia terms in system (6) gives the following conditions on Δt:

$$\Delta t \leqq \varepsilon \Delta x^2/4 \quad \text{and} \quad \Delta t \leqq R\,\Delta x^2/4 \tag{7a}$$

These are the von Neumann necessary conditions for stability.

It can be readily shown that these conditions are sufficient as well, since the amplification matrix G of the system above mentioned has a complete set of eigenvectors and the Gram determinant of the normalized eigenvectors of G are bounded away from zero, if condition (a) is satisfied (with the sole exception of $R=\varepsilon$), (see [11]).

It is concluded under the natural restriction of a Fourier type analysis that for highly viscous flows the stability is not perturbated by the presence of convective terms.

It is fair to say that the condition $\Delta t \leqq \varepsilon \Delta x^2/4$ is draconian if we notice that ε must be very small (in the limit $\varepsilon \to 0$ for evolutionary problems). In the case of steady state problems the value of ε could be in principle arbitrary since we are not interested in the intermediate solutions but in the final one.

Let us now consider the influence of diffusion and inertia terms; for that purpose we write down a simplified model of the system under study

$$\frac{\partial \omega}{\partial t} + \sum_{i=1}^{2} u_i \frac{\partial \omega}{\partial x_i} - \frac{1}{R}\Delta\omega = 0$$

and the difference analogue approaching it

$$W_{jk}^{n+1} - (1-4q)\,W_{jk}^n - (q-r_1)\,W_{j+1,k}^n - (r_1+q)\,W_{j-1,k}^n - (q-r_2)\,W_{j,k+1}^n - (r_2+q)\,W_{j,k-1}^n = 0 \tag{7b}$$

where $q = \Delta t/R\,\Delta x^2$, $r_i = U_i \Delta t/2\Delta x$, $\Delta x_1 = \Delta x_2 = \Delta x$.

It can be shown by a linear stability analysis that a necessary condition for stability (see [2] and [10]) gives

$$q \leqq \tfrac{1}{4} \quad \text{and} \quad r \leqq 4q \quad \text{where} \quad r = (|U_1| + |U_2|)\Delta t/\Delta x$$

or

$$\Delta t \leqq R\,\Delta x^2/4 \quad \text{and} \quad \Delta x \leqq 4/\{R(|U_1| + |U_2|)\}\,.$$

In practice we have used the following conditions easily derived from the formers

$$\Delta t \leqq R\,\Delta x^2/4 \quad \text{and} \quad \Delta t \leqq 4/R\,(|U_1| + |U_2|)^2\,.$$

Stability in the maximum norm can be easily established if in (7b) $1-4q \geqq 0$ and $q - |r_i| \geqq 0$. If these conditions are valid all the coefficients in (7b) are positive and their sum equals 1, stability follows.

These conditions can be written in the following form

$$\Delta t \leqq R\,\Delta x^2/4 \text{ and } \Delta x \leqq 2/(R\,|U_i|)\,.$$

It is interesting to observe that if

$$\Delta x = 2/(R|U_i|)$$

it follows that

$$\Delta t/\Delta x \leqq 1/(2|U_i|)\,,$$

which is a hyperbolic type stability condition.

Finally it is observed that a linear stability analysis in the absence of the term representing the physical viscosity shows that the difference analogue of the model equation is unconditionally unstable.

4. Formulation of the square cavity flow problem

We have chosen to test our method the problem of the steady flow of a viscous incompressible fluid in a square cavity ([5]).

The problem is formulated as follows (see [6]): Let the points (0, 0) (1,0) (1, 1) and (0, 1) be denoted by A, B, C, D respectively. Let Γ be the square whose vertices are A, B, C, D and denote its interior by Ω (see fig. 1).

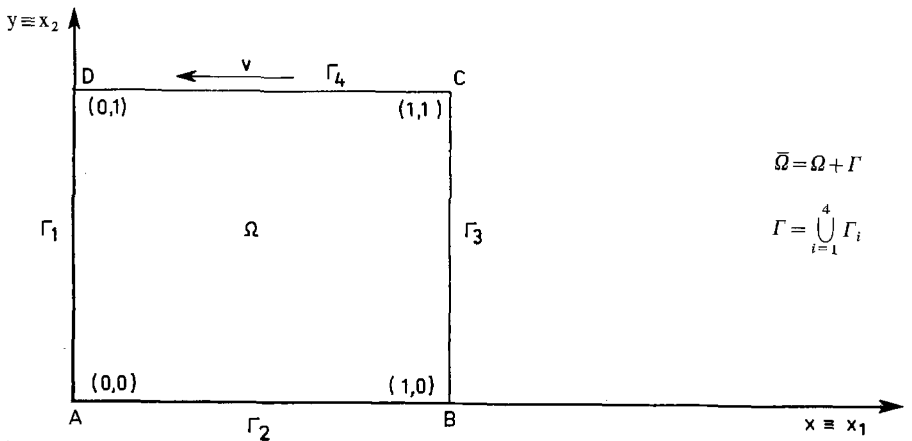


Figure 1

Let us consider the two-dimensional steady state transport vorticity equation to be satisfied on Ω

$$\sum_{i=1}^{2} u_i \frac{\partial\omega}{\partial x_i} = \frac{1}{R}\Delta\omega\,, \quad \Delta\varphi = -\omega\,, \qquad i = 1, 2\,, \tag{8}$$

where R is the Reynolds number: $R = vD/\nu$, reference velocity: v at the boundary DC, reference length: one side of the square, $u_1 = \partial\varphi/\partial x_2$, $u_2 = -\partial\varphi/\partial x_1$, φ the stream function and ω the vorticity.

The boundary conditions to be satisfied on Γ are:

$$\varphi(x) = \frac{\partial\varphi(x)}{\partial n} = 0 \qquad \text{for} \quad x \in \Gamma_1$$

$$\varphi(x) = \frac{\partial\varphi(x)}{\partial n} = 0 \qquad \text{for} \quad x \in \Gamma_2$$

$$\varphi(x) = \frac{\partial\varphi(x)}{\partial n} = 0 \qquad \text{for} \quad x \in \Gamma_3$$

$$\varphi(x) = 0, \quad \frac{\partial\varphi(x)}{\partial n} = -1 \quad \text{for} \quad x \in \Gamma_4$$

where **n** denotes the normal to the boundary.

We are going to approach problem (8) by (4) defined in G and satisfying (4) for:

$$x = \{x_1, x_2\} \in \Omega, \qquad 0 < t < T,$$

together with the initial conditions:

$$\omega(x, 0) = \omega_0(x), \quad x \in \Omega, \qquad \omega_0(x) \text{ arbitrary},$$
$$\varphi(x, 0) = \varphi_0(x), \quad x \in \Omega, \qquad \varphi_0(x) \text{ arbitrary}.$$

The boundary conditions coincide with those of problem (8), in general

$$\varphi(x, t) = \varphi(x, 0), \quad x \in \Gamma, \quad t > 0.$$

We remark that we consider here the solution of (4) when $t \to \infty$ and therefore ε is arbitrary, rather than $\varepsilon \to 0$ which would be the case for a real evolutionary problem.

5. Numerical treatment

In order to integrate (8) by means of the difference analogue given by (6) in the square cavity defined in section 4 we introduce inside it a net that we have chosen squared for simplicity reasons; the distance step is Δx. The number of intervals over one side is defined as $M = 1/\Delta x$. The interior of the cavity contains $(M-1)^2$ nodes, each of them indexed l, m corresponding to the coordinates

$$x = l\Delta x = x_l \qquad y = m \quad y = y_m \qquad l, m = 1, 2, \ldots M-1.$$

In order to discretize the boundary conditions appears necessary to extend the domain by means of a supplementrary row of nodes in each side of the exterior of it (see fig. 2).

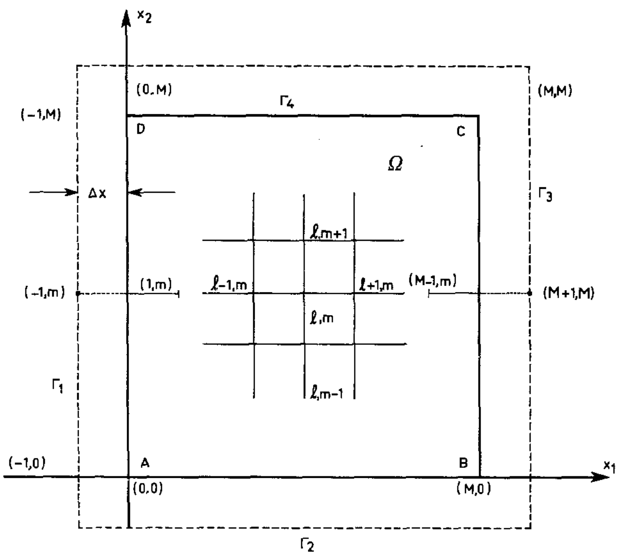


Figure 2

For the approximation of the derivatives we use centered differences formulae, for example

$$\left(\frac{\partial \varphi}{\partial x_1}\right)_{l,m} = \frac{1}{2\Delta x}\left(\varphi_{l+1,m} - \varphi_{l-1,m}\right) + O(\Delta x^2)$$

$$\left(\frac{\partial^2 \varphi}{\partial x_1^2}\right)_{l,m} = \frac{1}{\Delta x^2}\left(\varphi_{l+1,m} - 2\varphi_{l,m} + \varphi_{l-1,m}\right) + O(\Delta x^2).$$

The same is done with the x_2 axis.

The boundary conditions for φ are discretized by means of a centered formula for example on Γ_1 we have

$$\left(\frac{\partial\varphi}{\partial n}\right)_\Gamma = -\left(\frac{\partial\varphi}{\partial x_1}\right)_{\Gamma_1} = \frac{1}{2\Delta x}(\varphi_{-1,m} - \varphi_{1,m}) + O(\Delta x^2).$$

The same for the other boundaries.

In order to test the convergence of the difference analogue (6) when $t\to\infty$ to the solution of the steady state problem defined in (8) the following criterium is used

$$\underset{\omega,\varphi}{\text{Max}}\,\{\underset{l,m}{\text{Max}}\,|W_{l,m}^{n+1} - W_{l,m}^{n}|\} \leqq \delta$$

where δ is a small parameter in general varying from 10^{-4} to 10^{-7}.

In the following sections when speaking of precision we are referring to the criterium above-mentioned

6. Preliminary tests

Before attempting the solution of the square cavity flow problem defined in section 4 we have made some preliminary computations with a very simple problem in order to study the behaviour of the numerical scheme chosen. The analytical solution of this problem is known.

System (4) is now solved (in the absence of inertia terms) in the same domain defined in section 4 subjected to the following boundary conditions

$$\varphi(x,t) = x(1-x)^2 \quad \text{for} \quad x\in\Gamma_s, \quad s=2,4$$
$$\varphi(x,t) = 0 \quad \text{for} \quad x\in\Gamma_s, \quad s=1,3$$

The boundary values of the vorticity function are easily derived from the stream function with the aid of the second equation of system (4).

The initial conditions are

$$\varphi(x,0)=0,\ \omega(x,0)=0 \qquad x\in\Omega.$$

The analytical solution is

$$\varphi(x,t) = x(1-x)^2 \quad \text{for} \quad x\in\bar{\Omega}$$

We now define the term E as

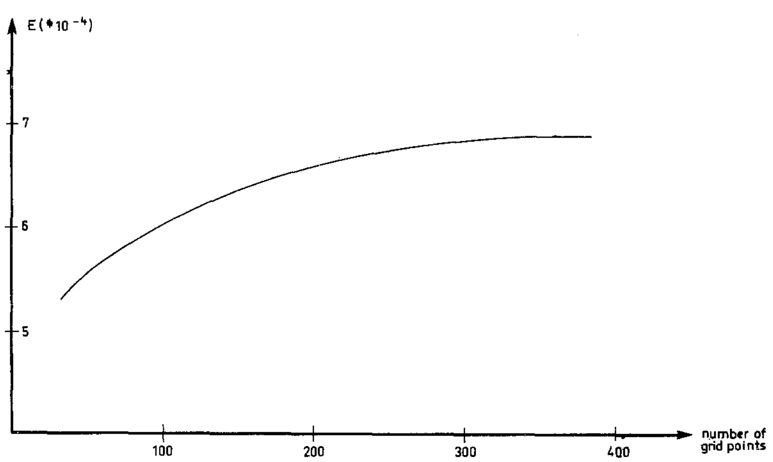


Figure 3. Behaviour of the term E when the grid is refined.

$$E = \underset{\omega,\varphi}{\mathrm{Max}} \left\{ \max_{l,m} |W_{\mathrm{anal}\,l,m} - W_{\mathrm{num}\,l,m}| \right\}$$

where $W_{\mathrm{anal}\,l,m}$ is the value of the analytical solution in the point l, m
$W_{\mathrm{num}\,l,m}$ is the value of the numerical solution in the point l, m
$W = \{\omega, \varphi\}$ is a vector valued function.

With the help of the test problem just defined the following questions have been studied:

(a) Behaviour of the term E when $n \to \infty$ for fixed values of Δt and Δx.
(b) Behaviour of the term E when Δt, $\Delta x \to 0$ for fixed $T = n \cdot \Delta t$.
(c) Behaviour of the term E when $\varepsilon \to 0$, under item (a).
(d) Behaviour of the term E under item (a) when different initial conditions were used.

In relation to point (a) Table 1 shows the values of E as a function of time for a grid size of 10×10 with $\Delta t/\Delta x^2 = 0.25$ and $\varepsilon = 1$.

TABLE 1

n° cycles	E
10	1.25
50	1.23×10^{-1}
100	1.00×10^{-2}
150	8.15×10^{-4}
200	6.63×10^{-5}
250	5.39×10^{-6}
291	7.25×10^{-7}

It is readily seen that the system converges to the steady state.

In relation to item (b) the following series of runs were made for fixed $T = n\Delta t = 390.6 \times 10^{-3}$, $\Delta t/\Delta x^2 = 0.25$ (see Table 2).

TABLE 2

Run	Grid size	$E\,(*10^{-4})$
1	8×8	5.30
2	12×12	6.12
3	15×15	6.52
4	20×20	6.92

The corresponding plotting is given in fig. (3). The behaviour of E as a function of the number of points (in space) although increasing, seems to reach an asymptotic value.

As already has been observed the variation of ε does not affect the final solution of a steady state problem but do affect the rate at which this solution is approached [3]. This can be shown in Table 3 in which for a fixed number of cycles, ratio $\Delta t/\Delta x^2$, grid size and the same initial conditions it was varied the value of ε. Parameters: n° cycles$\,=300$, $\Delta t/\Delta x^2 = 0.005$, grid size: 10×10.

TABLE 3

$\varepsilon = 1/c^2$	E
1/2	9.08×10^{-2}
1/5	3.71×10^{-2}
1/10	8.43×10^{-3}
1/50	$9{\cdot}11 \times 10^{-7}$

It is readily seen that the rate of convergence is improved with decreasing values of ε (greater values of c^2). In this case we have used the exact value of ω.

The role played by c^2 when it is increased may be seen as if we were using a greater time step in the second equation of system (6) *i.e.* $\Delta t_{\text{actual}} = c^2 \cdot \Delta t$.

In order to study the influence of the arbitrary initial conditions in relation to the total time needed to reach the steady state under a given precision ($\delta = 10^{-6}$) several runs were made with different values of W_0. It was found that in principle the total time elapsed was not radically influenced by the initial conditions chosen.

The major conclusions of this section are:

(a) The term E tends asymptotically to zero when the number of cycles is increased for fixed Δt and Δx. i.e. the solution converges to the steady state.

(b) The error between the analytic and the numerical solutions seems to reach an asymptotic value when the grid is refined (Δt, $\Delta x \to 0$).

(c) The term ε acts as a relaxation parameter and naturally the convergence is increased when epsilon decreases. The optimal value of ε is determined by stability criterium.

(d) The choice of the initial value is in principle arbitrary and do not affect appreciably the total computing time necessary to reach the steady state under a given precision δ.

7. Numerical computations

This section is devoted to the study of the solution of the square cavity flow problem defined in section 4, taking in account, in relation to the numerical method used, the preliminary considerations exposed in section 6. Further some details on the role of ε are given and finally the plottings of the variables under study are presented.

TABLE 4

R	Δx	Δt	*N° cycles*	*Precision* (δ)	*Total computing time (min)*	
0	1/10	$0.25 * \Delta x^2$	204	10^{-6}	1	ICL
0	1/10	$0.20 * \Delta x^2$	253	10^{-6}	1.25	ICL
10	1/10	$0.25 * \Delta x^2$	205	10^{-6}	1	ICL
10	1/10	$0.20 * \Delta x^2$	255	10^{-6}	1.25	ICL
50	1/10	$0.25 * \Delta x^2$	221	10^{-6}	1.1	ICL
50	1/10	$0.20 * \Delta x^2$	270	10^{-6}	1.35	ICL
100	1/10	$0.025 * \Delta x$	243	10^{-6}	1.2	ICL
100	1/10	$0.020 * \Delta x$	272	10^{-6}	1.35	ICL
100	1/10	$0.015 * \Delta x$	359	10^{-6}	1.80	ICL
200	1/10	$0.010 * \Delta x$	459	10^{-6}	2.3	ICL
400	1/20	$0.0025 * \Delta x$	3000	3×10^{-5}	47	ICL
400	1/20	$0.0025 * \Delta x$	5000	1.3×10^{-6}	18	IBM
400	1/40	$0.0025 * \Delta x$	6000	1.5×10^{-5}	97	IBM

The square cavity flow problem has been solved for different values of Reynolds number, grid size and precision. The calculations were carried out on the I.C.L. 1905 and afterwards on the I.B.M. 360/65 from the Delft University of Technology. The language used was Algol.

The most representative results are shown in Table 4. The same initial conditions were used for all the cases, i.e.

$$\varphi(x, 0) = 0 \quad \text{for} \quad x \in \Omega$$

$$\omega(x, 0) = 0 \quad \text{for} \quad x \in \Omega$$

The computations were ended when a preselected precision was reached or when the total computing time was overpassed.

We remind that $R=0$ indicates the absence of inertia terms.

To study the influence of the parameter ε for this specific problem we have made a series of runs with fixed values of the following parameters: grid size (10×10), $\Delta t/\Delta x^2=0.005$, Reynolds number $=10$, number of cycles: 300 and the same initial conditions in all the runs. In Table 5 are shown the results.

TABLE 5

$\varepsilon=1/c^2$	E
1/2	17.8×10^{-3}
1/5	12×10^{-3}
1/10	9.05×10^{-3}
1/50	7.31×10^{-3}

It is seen that the rate of convergence is improved with greater values of c^2.

We may now drop the following general conclusions.

(a) The convergence to the steady state is a function of the total time elapsed i.e. greater time steps obviously increases the convergence
(b) The maximum allowable time step is a function of the Reynolds number with the obvious consequence that the total computing time is also a function of the Reynolds number
(c) In the higher range of Reynolds number (>100) the maximum time step is inversely proportional to the Reynolds number
(d) The rate of convergence to the steady state is improved with decreasing values of the parameter ε which, in turn, brings a restriction on the time step.
(e) In the first cycles of the computation the convergence is very fast but the last figures for higher precision are hard to obtain.

In order to compare our results with those obtained in [4], [5], [6], [7] and [8] we have made some plottings in which is shown the development of the flow with different Reynolds numbers.

The first pair of figures (4 and 5) shows the equivorticity lines (Fig. 4) and the streamlines (Fig. 5) for $R=0$ in a grid size of 32×32 with a precision of 10^{-6}.

The next two figures (6 and 7) are the corresponding tridimensional plottings of the preceding experiments. Fig. 6 represents the streamfunction and fig. 7 the vorticity function. (Both figures are turned 90 degrees around the z-axis in relation to fig. 4).

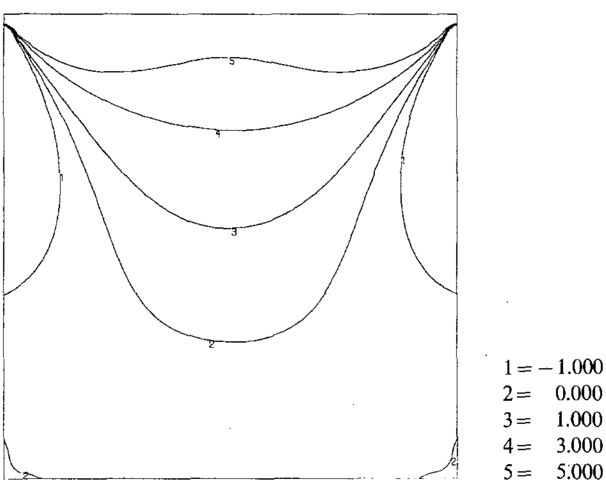


Figure 4. Equivorticity lines for $R=0$.

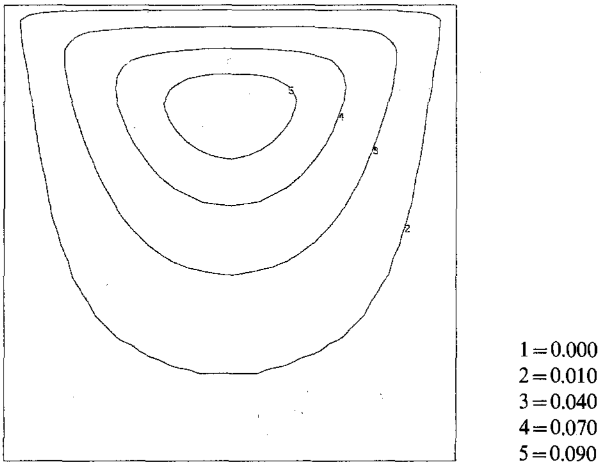


Figure 5. Streamlines for $R=0$.

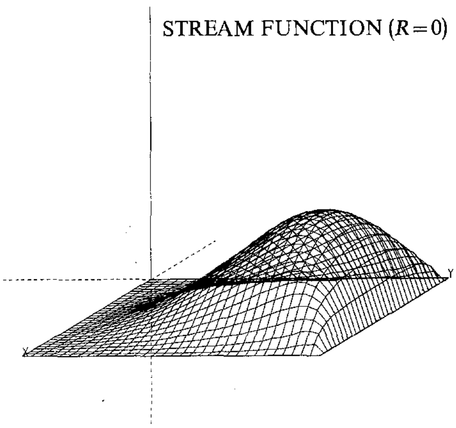


Figure 6

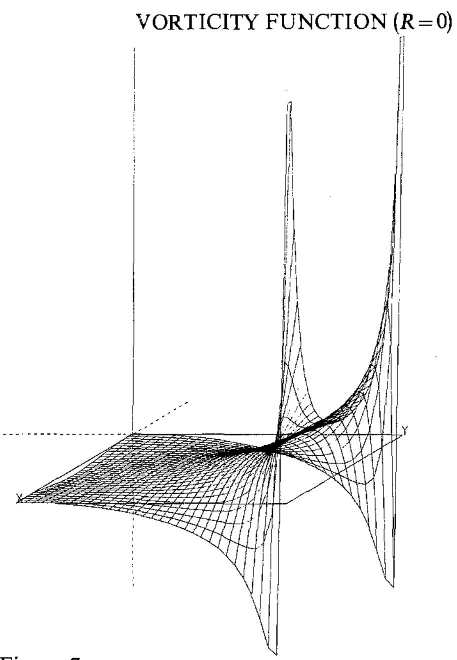


Figure 7

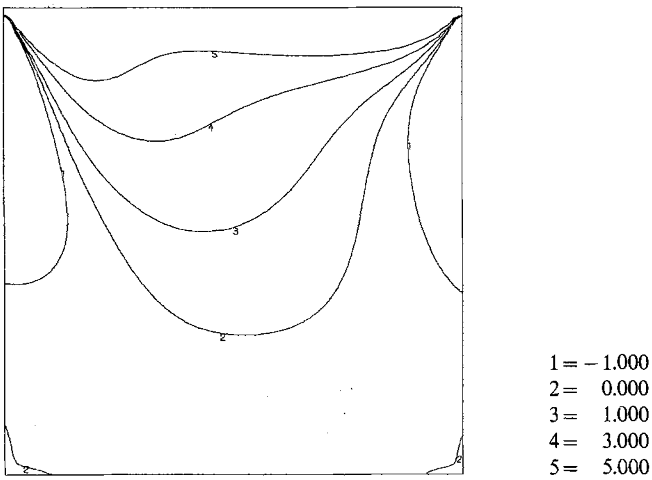


Figure 8. Equivorticity lines for $R=50$.

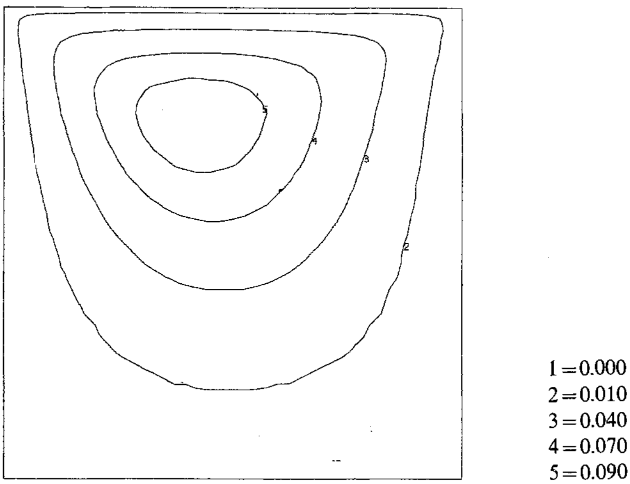


Figure 9. Streamlines for $R=50$.

The next pair of figures (8 and 9) shows the patterns of the flow for $R=50$ in a grid size of 40×40 with a precision of 10^{-6}. The corresponding tridimensional plottings are shown in figures 10 and 11. Fig. 12, 13, 14 and 15 show the same for $R=400$ in a grid size of 20×20 with a precision of 1.3×10^{-6}.

Figures 16, 17, 18 and 19 show the contour lines and perspective plotting of the vorticity and stream function. The parameters are: $R=400$, grid size 40×40 and precision equal to 1.5×10^{-5}.

It is observed that the streamlines are quite similar for different Reynolds numbers with the exception of a slight change in the position of the center of the vortex. On the contrary the vorticity is more influenced by increasing Reynolds numbers loosing the symmetric pattern shown at $R=0$ and suggesting that at very high Reynolds numbers will consist of a core of nearly uniform vorticity [5]. Two secondary eddy formations are observed at the bottom of the square cavity.

These rather general considerations on the results obtained as compared with those of the authors mentioned above show a good agreement specially in the lower range of Reynolds numbers.

For higher Reynolds numbers the flow develops very steep gradients near the walls (diminishing thickness of the boundary layer, [5]), neither this phenomena nor the formation of secondary eddies can be correctly described with coarse mesh sizes since in those cases a great

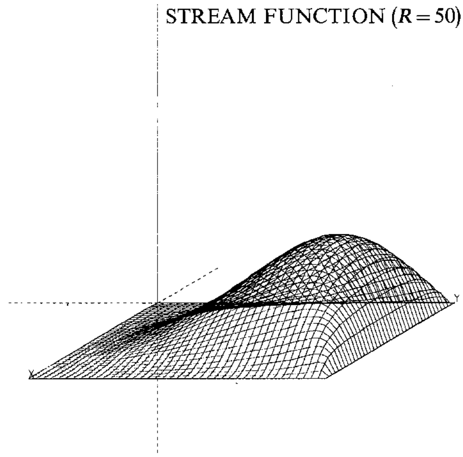


Figure 10

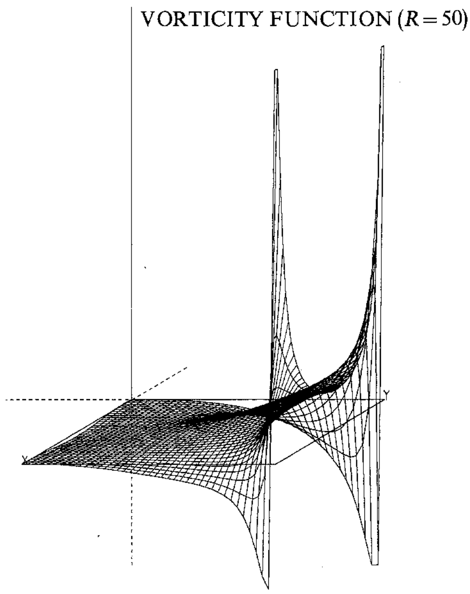


Figure 11

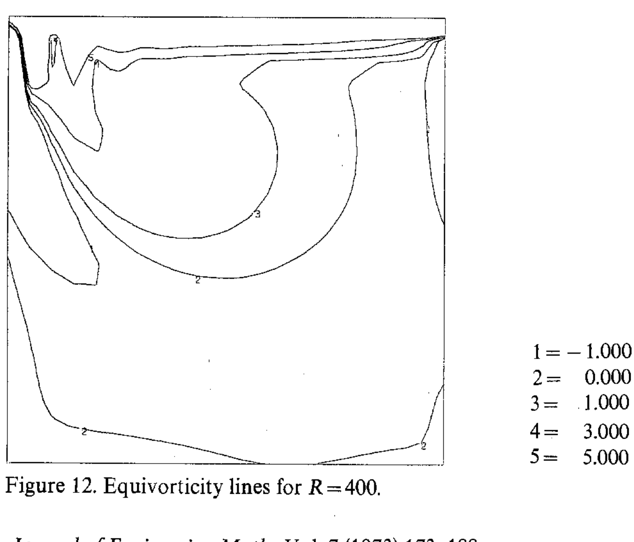


Figure 12. Equivorticity lines for $R=400$.

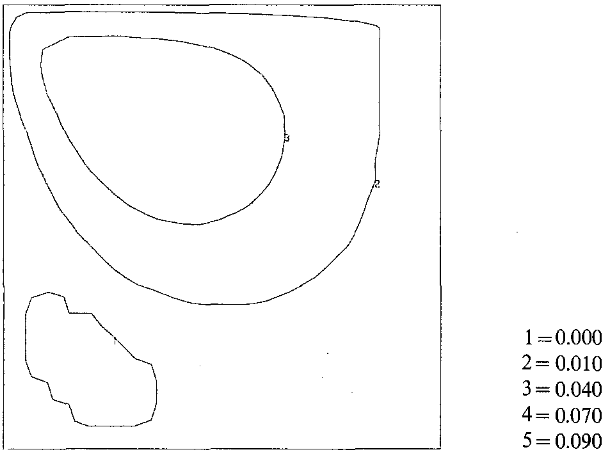


Figure 13. Streamlines for $R = 400$.

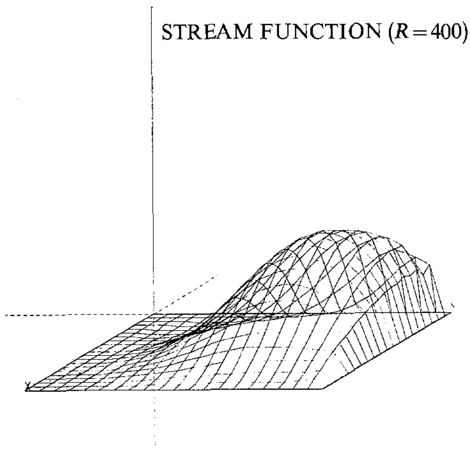


Figure 14

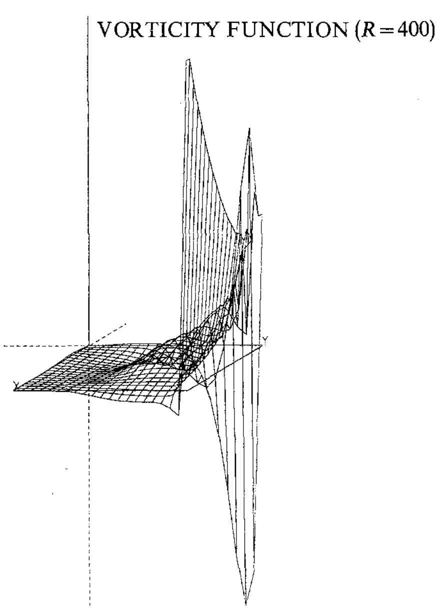
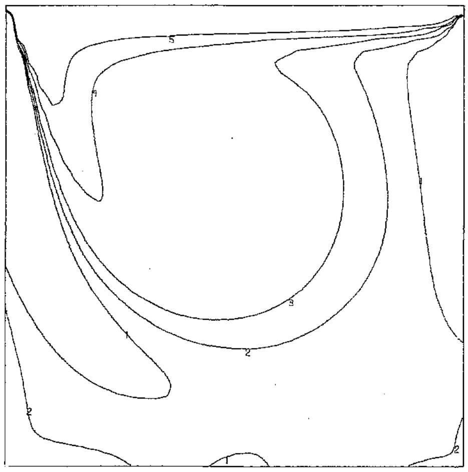
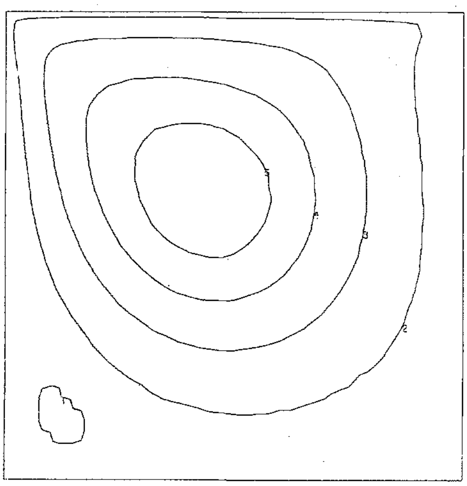


Figure 15



1 = − 1.000
2 = 0.000
3 = 1.000
4 = 3.000
5 = 5.000

Figure 16. Equivorticity lines for $R = 400$.



1 = 0.000
2 = 0.010
3 = 0.040
4 = 0.070
5 = 0.090

Figure 17. Streamlines for $R = 400$.

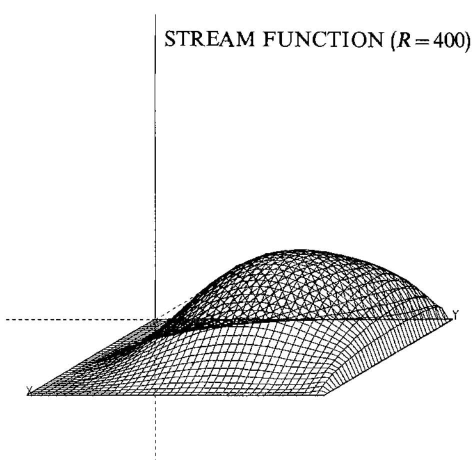


Figure 18

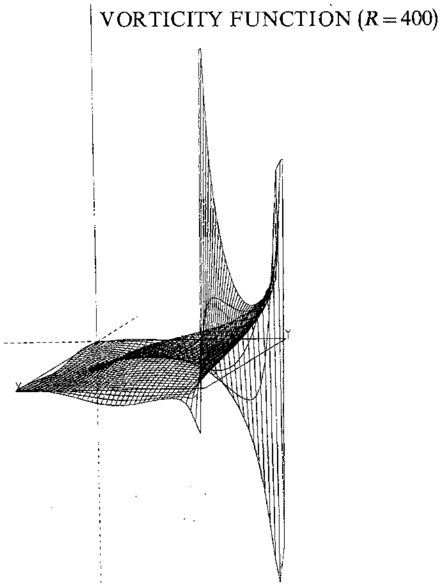


Figure 19

truncation error is involved. The strong oscillations observed in figure 15 can be explained by the coarseness of the grid used (20×20). These type of oscillations tend to disappear when the grid is refined (fig. 19, 40×40 grid size). But it is possible to observe in fig. 19 a small amplitude oscillation superimposed to the solution. We believe that this phenomena is caused by the numerical procedure used in the approximation of the convective terms (centered).

This effect might be diminished by the use of the so called "upstream" difference approximation. Unfortunately this method introduces a numerical diffusion which tends to overshadow the physical viscosity rendering under certain conditions meaningless calculations.

8. Conclusions

The method presented here is simple and relatively fast requiring a few preliminary tests for obtaining the optimum time step for a given accuracy and precision.

On the contrary the computing time is a direct function of the Reynolds number and drastically increases for Reynolds greater than 400. Therefore we are considering, for this range of Reynolds numbers, the possibility of using implicit schemes of the splitting type and a different treatment of the Poisson type equation.

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